Tutorial 9 2022.11.30

9.1 Supplementary problems in Assignment 11

Problem 9.1 Let S be the triangle with vertices at $(1,0,0),(0,2,0),(0,0,7)$ with normal pointing upward. Verify Stokes' theorem for the vector field $\mathbf{F} = x\mathbf{i} + 3z\mathbf{j}$.

Problem 9.2 Let S be the surface given by $(x,y) \mapsto (x,y,f(x,y)), (x,y) \in D$. That is, it is the graph of f over the region D. Show that in this case Stokes' theorem

$$\iint_S \nabla \times \mathbf{F} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

($\mathbf{F}$ is a smooth vector field on S) can be deduced from Green's theorem for some vector field on D. Hint: Let the boundary of D be $\mathbf{r}(t) = (x(t), y(t))$. Then the boundary of S is $\mathbf{c}(t) = (x(t), y(t), f(x(t), y(t)))$. Convert the integration in S and C to the integration on D and the boundary of D respectively.

9.2 A proof of Brower's fixed point theorem [1]

Let $D \subset \mathbb{R}^2$ be the unit disk $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. Then the boundary ∂D of D is the unit circle $\mathbb{S}^1 = \{(\cos\theta, \sin\theta) | \theta \in [0, 2\pi]\}$.

Theorem 9.1

Let $f : D \to D$ be a continuous map. There exists $x \in D$ such that $f(x) = x$.

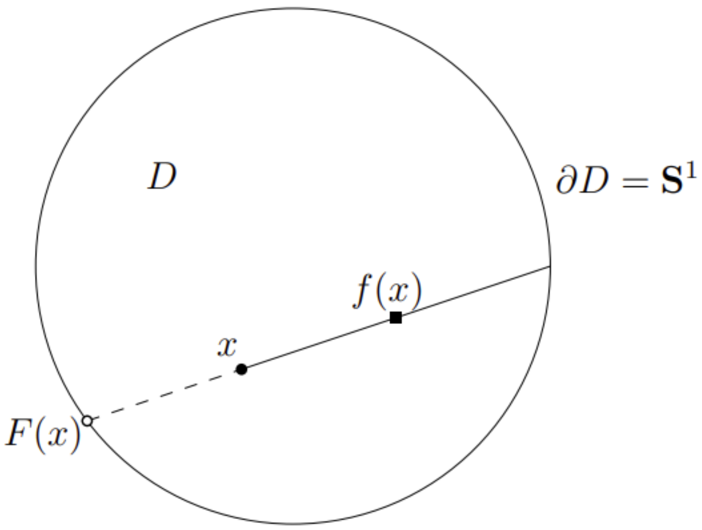

Figure 9.1

Proof We argue by contradiction. If for all $x \in D, f(x) \neq x$, consider a ray starting from $f(x)$ which passes through x after which it has exactly one intersection with ∂D. Let's denote the intersection point by $F(x)$. Therefore, we have a well-defined map $F : D \to \mathbf{S}^1 \subset \mathbb{R}^2, \quad x \mapsto F(x)$. The map F is continuous. We

[1] The reference for this proof is in page 595 of Pin Yu's mathematical analysis, and you could find the book on https://github.com/wuyudi/good-books

could further assume it is continuously differentiable (One will know why we can make this assumption from differential topology).

The map F could be written in terms of coordinates $F(x,y) = (u(x,y), v(x,y)) \in \mathbb{R}^2$. Then $u(x,y) = x, v(x,y) = y$ if (x,y) lies in the circle $\mathbb{S}^1$.

Consider the integration

$$\begin{aligned} I &= \oint_{\partial D} xdy - ydx \\ &= \int_D 2dxdy \\ &= 2\pi. \end{aligned}$$

On the other hand,

$$\begin{aligned} I &= \oint_{\partial D} xdy - ydx \\ &= \int_0^{2\pi} \cos\theta d(\sin\theta) - \sin\theta d(\cos\theta) \\ &= \int_0^{2\pi} u(\cos\theta, \sin\theta) d(v(\cos\theta, \sin\theta)) - v(\cos\theta, \sin\theta) d(u(\cos\theta, \sin\theta)) \\ &= \oint_{\partial D} u(v_x dx + v_y dy) - v(u_x dx + u_y dy) \\ &= \oint_{\partial D} (uv_x - vu_x)\, dx + (uv_y - vu_y)\, dy \end{aligned}$$

We write u_x for $\frac{\partial u}{\partial x}$,etc., for convenience.

Since the image of F lies in $\mathbb{S}^1$, we have

$$u^2 + v^2 = 1 \Rightarrow \begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \end{cases}$$

As $(u,v) \neq (0,0)$, the the determinant for the linear equations is zero,

$$u_x v_y = u_y v_x$$

Therefore,

$$\begin{aligned} I &= \oint_{\partial D} (uv_x - vu_x)\, dx + (uv_y - vu_y)\, dy \\ &= \oint_{\partial D} ((uv_y - vu_y)_x - (uv_x - vu_x)_y) dxdy \\ &= 0, \end{aligned}$$

which is a contradiction.